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Singularity spectra of strongly inhomogeneous multifractals

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Abstract. Generalized multifractal formalism is used to study singularity spectra of strongly inhomogeneous multifractals characterized by coarse-grained probability measures with zero minimal and/or infinite maximal Hölder exponents. Due to involving two additional types of scaling indices, the generalized formalism is shown to be able to describe complex multifractal objects by families of bivariate spectra rather than familiar single spectra of singularity strengths of one type, providing a more complete and adequate characteristics of such objects. It is proved that the families of extended singularity spectra can reveal unusual forms with many maxima, reflecting complex scaling structures of strongly inhomogeneous multifractals.

PACS. 05.50.+q Lattice theory and statistics (Ising, Potts, etc.) – 05.70.-a Thermodynamics – 64.10.+h General theory of equations of state and phase equilibria – 68.35.Rh hase transitions and critical phenomena

1 Introduction

Coarse-grained probability measures determined for complex objects undergo usually simple scaling laws $p_i \sim \ell_i^{\bar{\alpha}_i}$, $i = 1, 2, ...$, with ℓ_i being the length scale of ith piece of a given object and $\bar{\alpha}_i$ denoting the respective Hölder exponent [1]. In various contexts, there appear, however, multifractal objects, for which $\bar{\alpha}_{\min} = 0$ and/or $\bar{\alpha}_{\max} = \infty$ as $\max \ell_i \to 0$. Evidently, when applying the traditional multifractal formalism [1] to complex measures with $\bar{\alpha}_{\max} = \infty$, one obtains singularity spectra $f(\bar{\alpha})$ of an unusual, left-sided shape [2,3] (no matter whether the space partitioning is uniform or not). The disappearance and/or the divergence of some values of Hölder exponents as the limit $\max \ell_i \to 0$ is approached suggest that local probability measures scale according to $p_i \sim \ell_i^{\alpha_i w_i(\ell_i)}$, where α_i, $i = 1, 2, ...$, denote scale-independent singularity strengths, $w_i(\ell_i)$ are functions of ℓ_i, such that $0 \leq w_i(\ell_i) \leq \infty$ and $\max w_i(0)/\min w_i(0) = \infty$. Then, within the conventional multifractal formalism, all singularities α_i associated with the set $\{w_i(0) = 0\}$ are converted to a single value $\bar{\alpha} = 0$ and/or all singularities α_i associated with the set $\{w_i(0) = \infty\}$ are converted a value $\bar{\alpha} = \infty$. Thus, in cases of strongly inhomogeneous measures, the traditional multifractal formalism cannot characterize properly entire sets of singularity exponents α_i. It has recently been proved that, if such measures are supported by multifractal sets, then these sets can adequately be characterized by introducing, in general, two additional kinds of scaling exponents [4]. Accordingly, the strongly inhomogeneous local measures can then be assumed to satisfy the relation $p_i \sim \ell^{\,\alpha_i \beta_i \gamma_i}$ with β_i and γ_i being two additional indices and $\ell = \min \ell_i^{\max w_i(\ell_i)}$. The new scaling exponents are defined as $\beta_i = \lim_{\ell_i \to 0} [\, w_i(\ell_i)/\max w_i(\ell_i)]$ and $\gamma_i = \lim_{\ell_i \to 0} [\, \max\{w_i(\ell_i)\} \ln \ell_i / \ln \ell\,]$. Clearly, these singularity strengths are well defined only if the respective limits exist as $\ell \to 0$. Consequently, the exponents α_i can be interpreted as strengths of inhomogeneities of local probabilities p_i with respect to scales $\ell_i^{w_i(\ell_i)}$. The indices β_i take values from a unit interval $[0, 1]$ and reflect the variability of w_i with regard to $\max w_i$. Finally, the exponents γ_i describe the variability of ℓ_i, *i.e.*, the they describe the nonuniformity of covering of supports of probability measures. Complex structures of underlying multifractal sets (*i.e.*, supports of probability measures characterized by particular triples of scaling exponents α_i, β_i, and γ_i) can be investigated using an extended multifractal formalism, introduced recently [4]. In general, this formalism involves singularity strengths of the three types, in contrast to the traditional formalism, which uses only one type of singularities.

In the present paper, the generalized formalism is applied to study singularity spectra of a probability measure being a superposition of binomial submeasures and the Gibbs distribution of energy levels of one-dimensional Ising model. Both the measures are associated with strongly inhomogeneous objects (in the sense specified above), but the first measure is constructed using a nonuniform space partition, while the second one is determined for a uniform coarsegraining. Thus, for the latter measure, the exponents γ_i are all equal, and, in this case, the generalized formalism involves only scaling indices α_i and β_i. It will be shown that singularity spectra determined for strongly inhomogeneous objects can have not a familiar simple concave form and can even display many maxima.

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2 Generalized singularity spectra

The extension of the conventional multifractal formalism consists in using two additional filtering variables, conjugated to the scaling exponents β_i and γ_i [4]. The corresponding generalized partition function is given by

$$\Gamma_n(q,r,s) = \sum_{i=1}^{n} p_i^q \, \ell^{\beta_i\gamma_i r + \gamma_i s} \tag{1}$$

with the variables $q, r, s \in (-\infty, \infty)$ and $\ell = \min \bar{\ell}_i$, where $\bar{\ell}_i = \ell_i^{\max w_i(\ell_i)}$. As $\ell \to 0$ (and $n \to \infty$), the function (1) is expected to satisfy the scaling relation $\Gamma_n(q,r,s) \sim \ell^{\sigma(q,r,s)}$, while the number of space boxes for which local probabilities are described by indices $\alpha_i \in [\alpha, \alpha + \mathrm{d}\alpha]$, $\beta_i \in [\beta, \beta + \mathrm{d}\beta]$, and $\gamma_i \in [\gamma, \gamma + \mathrm{d}\gamma]$ is assumed to scale as $N(\alpha,\beta,\gamma) \sim \ell^{-g(\alpha,\beta,\gamma)}$. Then, making use of the triple Legendre transformation [4], one has

$$\sigma(q,r,s) = \min_{\alpha,\beta,\gamma} [\, \psi_{q,r,s}(\alpha,\beta,\gamma)\,], \tag{2}$$

$$\psi_{q,r,s}(\alpha,\beta,\gamma) = \alpha\beta\gamma q + \beta\gamma r + \gamma s - g(\alpha,\beta,\gamma), \tag{3}$$

$$\{\alpha\}\{\beta\}\{\gamma\} = \partial_q \sigma(q,r,s), \tag{4}$$

$$\{\beta\}\{\gamma\} = \partial_r \sigma(q,r,s), \tag{5}$$

$$\{\gamma\} = \partial_s \sigma(q,r,s), \tag{6}$$

where $\partial_x = \partial/\partial x$, and $\{\alpha\}$, $\{\beta\}$, $\{\gamma\}$ symbolize functions $\alpha(q,r,s)$, $\beta(q,r,s)$, $\gamma(q,r,s)$, respectively, such that they fulfill the condition (2).

The generalized singularity spectra $f(\alpha,\beta,\gamma)$ are defined by the scaling relation $N(\alpha,\beta,\gamma) \sim \tilde{\ell}^{-f(\alpha,\beta,\gamma)}$ with $\tilde{\ell} = \min \ell_i$. Thus, the spectra f are related to the function g by

$$f(\alpha,\beta,\gamma) = \frac{\ln \ell}{\ln \tilde{\ell}}\, g(\alpha,\beta,\gamma). \tag{7}$$

Since $\ln \ell / \ln \tilde{\ell} \to 0$ as $\ell \to 0$, the function $g(\alpha,\beta,\gamma)$ must tend to zero (for all values of the exponents α, β, and γ) when $\ell \to 0$, in order to the singularity spectra $f(\alpha,\beta,\gamma)$ could remain finite in the limit $\ell \to 0$. This does not mean, however, that $\psi_{q,r,s}(\alpha,\beta,\gamma)$ and thereby $\sigma(q,r,s)$ depend on ℓ as the limit $\ell \to 0$ is approached. Indeed, as $\ell \to 0$, these quantities are dominated by respective, ℓ-independent products of scaling exponents and filtering variables.

The use of the above approach is based on determining the generalized partition function (Eq. (1)). Essentially, this quantity can be calculated when the functions $w_i(\ell_i)$, $i = 1, 2, ...$, or at least their values at particular scales ℓ_i, $i = 1, 2, ...$, are known. In cases when scaling properties of strongly inhomogeneous measures are not explicitly known (*i.e.* in cases of measures obtained experimentally), the functions $w_i(\ell_i)$ as well as the exponents α_i, β_i, and γ_i cannot be determined in a unique manner, as they can only be obtained with an accuracy to multiplicative constants. (Clearly, products $\alpha_i w_i(\ell_i)$ are defined in a unique way.) However, the arbitrariness in choosing the multiplicative constants can be removed by adopting a simple, general method of determining the functions $w_i(\ell_i)$ (then, the exponents α_i, β_i, and γ_i are fixed). In the case when $\bar{\alpha}_{\max} = \infty$, it starts with finding the $\max w_i(\ell_i)$. In general, functions $w_i(\ell_i)$ and exponents α_i cannot be obtained by investigating probability measures p_i at different partition levels, since space coverings for these measures are redefined as the partition stage changes. Nevertheless, one can assume that the singularity α_i associated with $\min p_i$ (*i.e.*, with the maximal singularity connected to $\max w_i(\ell_i)$) does not change significantly (saturates) as the partitioning level increases. Then, one can determine the product $\max[\alpha_i w_i(\ell_i)]$ at different, high partition levels, and the $\max w_i(\ell_i)$ as well as the corresponding singularity $\max \alpha_i$ can be obtained using the condition that $\max w_i(1) = 1$. Consequently, for cases of $\bar{\alpha}_{\max} = \infty$, the remaining functions $w_i(\ell_i)$ can be expressed as

$$w_i(\ell_i) = [\max w_i(\ell_i)]^{x_i(\ell_i)} \tag{8}$$

with $0 \le x_i \le 1$. For small ℓ_i, the index $x_i(\ell_i)$ is approximately given by

$$x_i \approx \frac{\ln|\ln p_i| - \ln|\ln(\ell_i)|}{\ln[\max w_i(\ell_i)]}. \tag{9}$$

Clearly, this relation is not exact due to a contribution of α_i to p_i, as well as due to the existence of a constant factor c in the scaling law $p_i = c\ell_i^{\alpha_i w_i(\ell_i)}$. It should be pointed out that some of the functions $w_i(\ell_i)$, $i = 1, 2, ...$, can be identical, and then a given function $w_i(\ell_i)$ can be connected with different singularity strengths. Contributions of these singularities to $x_i(\ell_i)$ can roughly be eliminated by approximating $x_i(\ell_i)$ for a given space covering by discrete values:

$$x_i(\ell_i) \approx r_i \epsilon \tag{10}$$

with r_i being a natural number, such that $x_i(\ell_i) \ge r_i\epsilon$, $x_i(\ell_i) < (r_i + 1)\epsilon$, and with ϵ given formally by

$$\epsilon \approx \frac{\ln(\max \alpha_i) - \ln(\min \alpha_i)}{\ln[\max w_i(\ell_i)]}$$

where $\max \alpha_i$ and $\min \alpha_i$ are maximal and minimal, respectively, values of α_i associated with $w_i(\ell_i)$. It follows from the last relation that $\epsilon \ll 1$ for small ℓ_i, $i = 1, 2, ...$. In turn, when $\alpha_{\min} = 0$, the function $\min w_i(\ell_i)$ can be expressed as [3]

$$\min w_i(\ell_i) = -\frac{\ln|\ln \ell_i|}{\ln \ell_i}. \tag{11}$$

If additionally $\bar{\alpha}_{\max} < \infty$, the remaining functions $w_i(\ell_i)$ can be written now in the form:

$$w_i(\ell_i) = [\min w_i(\ell_i)]^{y_i(\ell_i)}, \tag{12}$$

where $0 \le y_i(\ell_i) \le 1$. When $\ell_i \to 0$, one obtains

$$y_i(\ell_i) \approx \frac{\ln|\ln p_i| - \ln|\ln \ell_i|}{\ln[\min w_i(\ell_i)]}. \tag{13}$$

By analogy with (10), this index can be estimated for a given ℓ_i using the relation

$$y_i(\ell_i) \approx r'_i \epsilon', \quad (14)$$

where r'_i denotes a natural number, such that $y_i(\ell_i) \geq r'_i\epsilon'$, $y_i(\ell_i) < (r'_i + 1)\epsilon'$, and ϵ' is given by

$$\epsilon' \approx \frac{\ln(\max \alpha_i) - \ln(\min \alpha_i)}{\ln[\min w_i(\ell_i)]}.$$

Note that $\epsilon' \ll 1$ for small ℓ_i, $i = 1, 2, ...$ Consequently, functions $w_i(\ell_i)$, $i = 1, 2, ...,$ can be found using equations (8–10) in the case of $\bar{\alpha}_{\max} = \infty$, or using equations (11–14) in the case of $\bar{\alpha}_{\min} = 0$, $\bar{\alpha}_{\max} = \infty$, with suitably chosen increments ϵ or ϵ', respectively. Obviously, any reliable results obtained for a given space covering should not vary strongly as ϵ or ϵ' change. This condition enables one to adjust the most appropriate values of ϵ and ϵ'.

It should be noted that, within the conventional multifractal formalism, the one-dimensional counterpart $\psi_q(\alpha)$ of the function $\psi_{q,r,s}(\alpha, \beta, \gamma)$ is assumed usually to be convex (downward) in α for all values of the filtering variable q. Indeed, the function $\psi_q(\alpha)$ has proved to be convex for many probability measures and, even, its convexity property has explicitly been shown for binomial measures [1]. However, there exist multifractal objects for which $\psi_q(\alpha)$ is not convex for all q, and, thereby, the resulting singularity spectra are not concave for all values of α (*cf.* Ref. [5]). In general, the function $\psi_q(\alpha)$ can have many minima, which can become one after the other an absolute minimum as q varies. Then, the position of the absolute minimum (*i.e.*, the value of α at which this minimum occurs) can change discontinuously, indicating the existence of phase transitions. Accordingly, the singularity spectra can possess many maxima and can reveal forms of envelopes of simple, concave functions.

As will be seen below, the function $\psi_{q,r,s}(\alpha, \beta, \gamma)$ determined for strongly inhomogeneous measures can also exhibit complex shapes with many minima. In such cases, the minimum in equation (1) denotes an absolute minimum, whose location changes continuously as q, r, s vary within some regions in the space of these variables, and change in a discontinuous way as borders of the regions are crossed.

3 Superpositions of multifractal measures

Define a probability measure $\mu = \sum_{k=1}^{m} \mu(k)$ with the submeasures $\mu(k)$ constructed by using a multiplicative process with probability rescalings $p_{1,k} = p_{2,k} = 1/2^k$, $k = 1, 2, ..., m$, and with corresponding length rescalings $\ell' = 1/3$, $\ell'' = 1/9$ (identical for all submeasures). Consider now a measure μ', defined as a normalized measure μ. Recent studies of multiscaling properties of the superposition of these binomial submeasures have been shown that the measure μ' reveal strongly inhomogeneous character, with $\bar{\alpha} \to \infty$ as $\ell \to 0$ [4]. The generalized partition function for μ' can written as (*cf.* Ref. [4])

$$\Gamma_{m,n}(q, r, s) = \sum_{k=1}^{m} \sum_{i=0}^{n} \binom{n}{i} p_k^q \, \ell^{\beta_k \gamma_i r + \gamma_i s}, \quad (15)$$

where $p_k = c_{m,n} 2^{-kn}$, $k = 1, 2, ..., m$, denote local probabilities, normalized at each partition stage n (the same for all submeasures $\mu(k)$), with $c_{m,n} = \left(1 - \frac{1}{2^n}\right) / \left(1 - \frac{1}{2^{mn}}\right)$ being the normalization constant, and the length scale $\ell = 9^{-mn}$. Owing to a special definition of the measure μ', all local probabilities are identical for each of the submeasures $\mu(k)$, but probabilities associated with different submeasures are different ($p_k \neq p_{k'}$ if $k \neq k'$). However, at a given partitioning level n, each of the submeasures $\mu(k)$, $k = 1, 2, ..., m$ is supported by 2^n segments of sizes ℓ_i, $i = 0, 1, ..., n$ (note that there exist only $n + 1$ different length scales). It should be pointed out that, although each of the submeasures $\mu(k)$ is uniform at each of the partitioning process, singularity strengths associated with local probabilities p_k, $k = 1, 2, ...,$ are, in general, different, due to nonuniformity of the support of each of the submeasures $\mu(k)$. The functions w_i introduced to describe strong measure inhomogeneities are in this case functions of k rather than functions of ℓ_i, and are all identical for each of the submeasures, *i.e.*, $w_i = w(k)$ for $i = 0, 1, ..., n$. Consequently, the scaling relation for the local probability assigned to ith segment belonging to the support of $\mu(k)$ can be expressed as $p_k \sim \ell_i^{\alpha_{k,i} w(k)}$ with $\ell_i = \left(\frac{1}{3}\right)^{n+i}$. The function $w(k)$ can be determined by assuming that $w(1) = 1$, and by assuming that the exponents $\alpha_{k,i}$ are independent of k in the limit $n \to \infty$ (this guarantees that the range of values of $\alpha_{k,i}$ remains nonzero and finite when m is a function (increasing) of n, and when $n \to \infty$). The above conditions implies that $w(k) = k$. Then, as can easily be verified, the set of exponents $\alpha_{k,i}$ becomes indeed identical for all submeasures $\mu(k)$ as $n \to \infty$, the indices β_k are the same for all i and for all construction stages of a given submeasure, while the set of singularities γ_i is identical for every measure $\mu(k)$. Thus, for large m and n, local scaling properties of the measure μ can be described by triples of exponents $\alpha_{k,i}$, β_k, and γ_i, $k = 1, 2, ..., m$, $i = 0, 1, ..., n$.

According to (7), the singularity spectra for the measure μ are given by

$$f(\alpha, \beta, \gamma) = m \, g(\alpha, \beta, \gamma). \quad (16)$$

These spectra can easily be investigated by changing one or two of the filtering variables and by keeping remaining variables or variable constant. For particular values of the filtering variables or variable, one then obtains families of multifractal spectra. Using (2–7, 15), and (16), the spectra $f(\alpha, \beta, \gamma)$ have been determined for $m = 10$ and $n = 10$. They are shown in Figure 1 as functions of α, for varying q, for r taking various constant (for each spectrum) values, and for $s = 0$. It is seen that, for each r, f is a concave function of α, and that the spectra f obtained for various r are enclosed between two concave envelopes. The bottom envelope is determined at the same minimal and maximal values of q ($q_{\min} \ll 0$, $q_{\max} \gg 0$), for all curves.

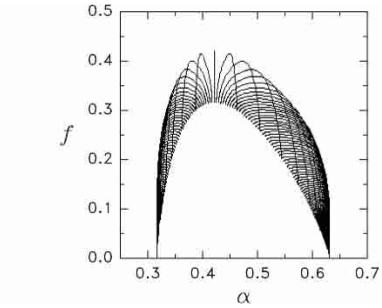

Fig. 1. Singularity spectra $f(\alpha, \beta, \gamma)$ *vs.* α for $s = 0$ and for various values of r, ranged from -4 (on the left) to 4 (on the right), with a step $\Delta r = 0.05$.

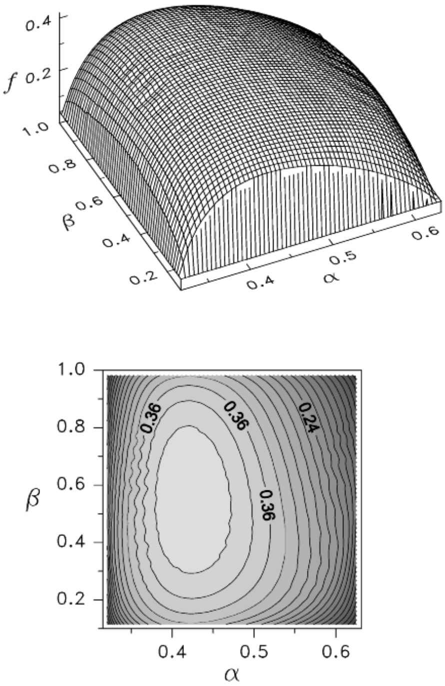

Fig. 2. Singularity spectra $f(\alpha, \beta, \gamma)$ and their contour plot *vs.* α and β for $s = 0$.

It turns out that minimal values of f obtained for $s = 0$ and for each r saturate quickly as $q \to -\infty$ and $q \to \infty$. The singularity spectra prove also to be concave functions of any two exponents (from the triple α, β, γ), with any of the filtering variables q, r, s being constant, except for spectra f considered as functions of β and γ with s kept constant. Dependences of some spectra on pairs of scaling exponents are illustrated in Figures 2–6. Note that irregularities of surfaces plotted in Figures 2, 4, and 5 are results of numerical approximations and do not correspond to any real effects. It is remarkable that, despite of the existence of abrupt changes (first-order phase transitions) in dependences of exponents α, β, γ on filtering variables q, r, s [4], the spectra f, treated as functions of two of the triple indices, are mostly concave for all values of respective pairs of scaling indices. Obviously, in the case of the surface shown in Figure 6, the function $\psi_{q,r,s}(\alpha, \beta, \gamma)$ possesses for some ranges of q and r more than one minimum, and the corresponding singularity spectra are not concave for all values of β and γ.

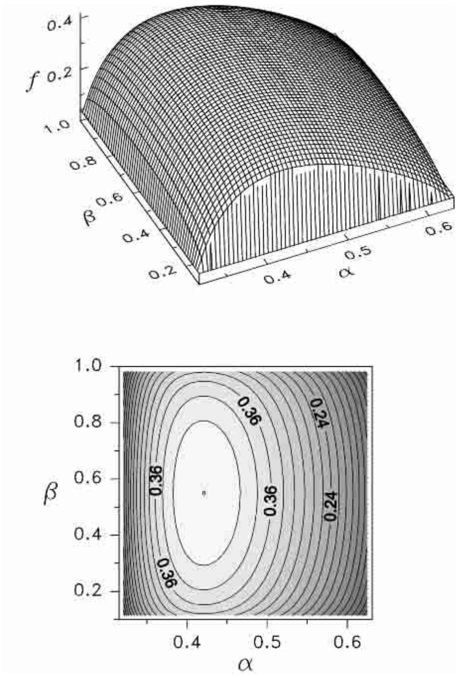

Fig. 3. Singularity spectra $f(\alpha, \beta, \gamma)$ and their contour plot *vs.* α and β for $r = 0$.

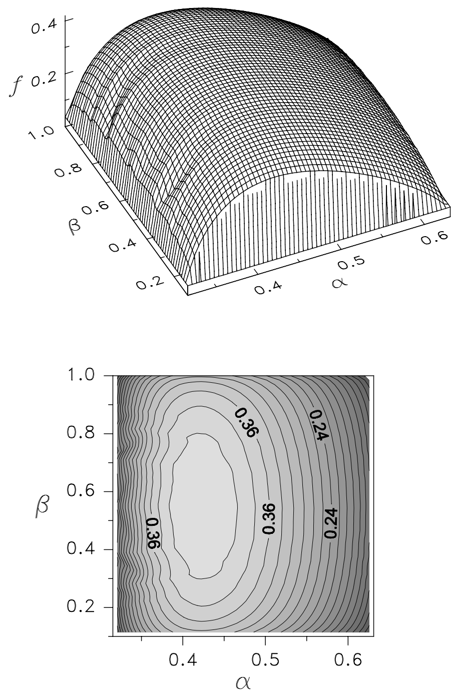

Fig. 4. Singularity spectra $f(\alpha, \beta, \gamma)$ and their contour plot *vs.* α and β for $q = 0$.

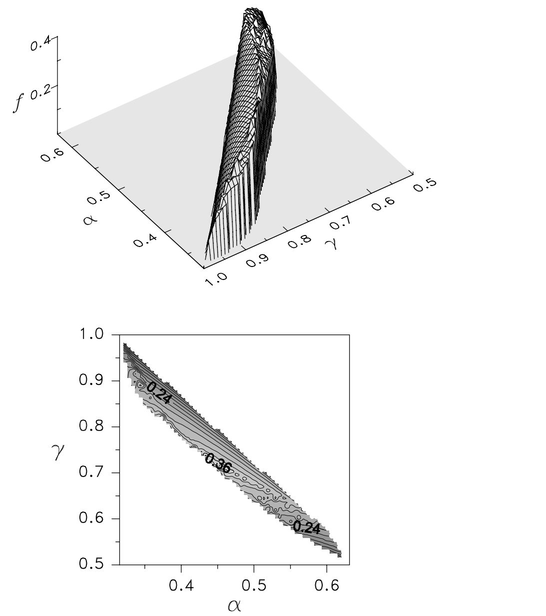

Fig. 5. Singularity spectra $f(\alpha, \beta, \gamma)$ and their contour plot *vs.* α and γ for $s = 0$.

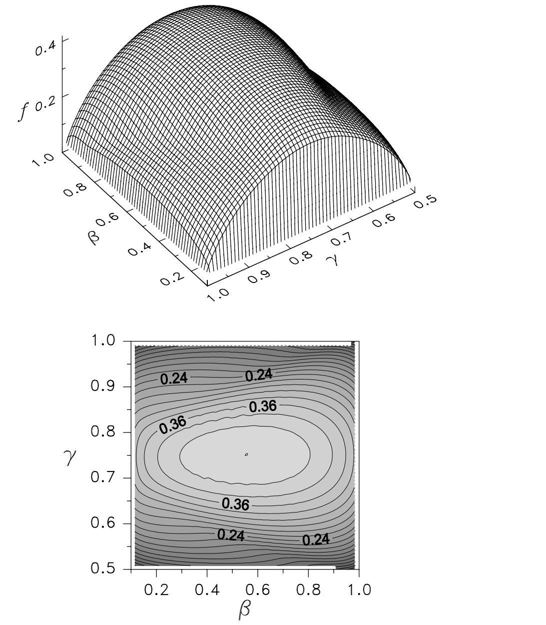

Fig. 6. Singularity spectra $f(\alpha, \beta, \gamma)$ and their contour plot *vs.* β and γ for $s = 0$.

4 Multifractal measures for one-dimensional Ising system

It has been proved that Gibbs distributions determined for Ising systems can be treated for all nonzero temperatures as multifractal measures, supported by discrete energy spectra [6]. In the case of the zero-field one-dimensional Ising model with periodic boundary conditions, the Gibbs distribution is determined by [4]

$$p_i^{(n)}(K) = \frac{1}{Z_n(K)} \binom{n}{2i} \mathrm{e}^{(n-4i)K}, \;\; i = 0, 1, ..., M_n - 1, \tag{17}$$

where $K = J/k_B T$ is the reduced coupling (with J being the nearest-neighbor interaction and T denoting temperature), n is the total number of spins, $Z_n(K) = \sum_{i=1}^{M_n} p_i^{(n)}(K)$ denotes the partition function, and $M_n = n/2 + 1$ is the total number of energy levels. As has been shown, the probabilities (17) display complex scaling properties with $\bar{\alpha}_{\max} \to \infty$ as $n \to \infty$ [7]. Since, for Ising systems, length scales are all equal [6], *i.e.*, $\ell_i = 1/M_n$, $i = 1, 2, ..., M_n$, and thereby the singularities γ_i are all identical [4], scaling properties of the probabilities (12) can completely be characterized by two kinds of exponents α_i and β_i. Accordingly, the generalized partition function for the one-dimensional Ising model can be written as

$$\Gamma_n(q, r) = \sum_{i=1}^{M_n} [\, p_i^{(n)}(K)]^q \, \ell^{\,\beta_i r} \tag{18}$$

with $\ell = \mathrm{e}^{-n}$. To determine the indices α_i and β_i, one has to find probability scales ℓ_i, determined by the relation $p_i^{(n)}(K) \sim \ell_i^{\alpha_i}$. This can be done by applying the numerical procedure described in Section 2. Then, the exponents α and β can be expressed as functions of q, and r, using a reduced version of equations (2–6), *i.e.*, by applying the double Legendre transformation:

$$\sigma(q, r) = \min_{\alpha,\beta} [\, \psi_{q,r}(\alpha, \beta)\,], \tag{19}$$

$$\psi_{q,r}(\alpha, \beta) = \alpha\beta q + \beta r - g(\alpha, \beta), \tag{20}$$

$$\{\alpha\}\{\beta\} = \partial_q \sigma(q, r), \tag{21}$$

$$\{\beta\} = \partial_r \sigma(q, r), \tag{22}$$

where $\sigma(q, r)$ and $g(\alpha, \beta)$ are given be the relations $\Gamma(q, r) \sim \ell^{\,\sigma(q,r)}$ and $N(\alpha, \beta) \sim \ell^{\,-g(\alpha,\beta)}$, respectively, with $N(\alpha, \beta)$ being the number of pairs of indices $\alpha_i \in [\alpha, \alpha + \mathrm{d}\alpha]$, $\beta_i \in [\beta, \beta + \mathrm{d}\beta]$, and $\{\alpha\}$, $\{\beta\}$ denote functions $\alpha(q, r)$, $\beta(q, r)$, satisfying the condition (19). By virtue of (7), the singularity spectra for the measure (17) are determined by

$$f(\alpha, \beta) = \frac{n}{\ln M_n} \, g(\alpha, \beta). \tag{23}$$

In Figure 7, shown are the spectra $f(\alpha, \beta)$ as functions of α, for particular values of r, while in Figure 8, the spectra are plotted *versus* α and β. As it is seen, these spectra display rather complicated forms with many maxima. It follows that the function $\psi_{q,r}(\alpha, \beta)$ has many minima, and that the minimum in the condition (19) refers to an absolute minimum for given q and r. It should be noted that, contrary to the measure consisting of binomial submeasures, in the case of the Gibbs measure (17), first-order phase transitions occurring in dependences of the scaling exponents on filtering variables [4], are reflected in vanishing of $f(\alpha, \beta)$ at α_c and β_c, such that $\beta_{\min} < \beta_c < \beta_{\max}$ and $\alpha_{\min} < \alpha_c < \alpha_{\max}$.

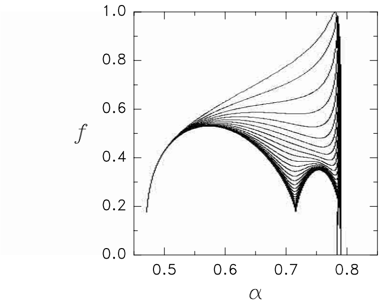

Fig. 7. Singularity spectra $f(\alpha, \beta)$ *vs.* α for $n = 100$ and for different, constant values of r, ranged from 0 (top) to 50 (bottom), with a step $\Delta r = 1$.

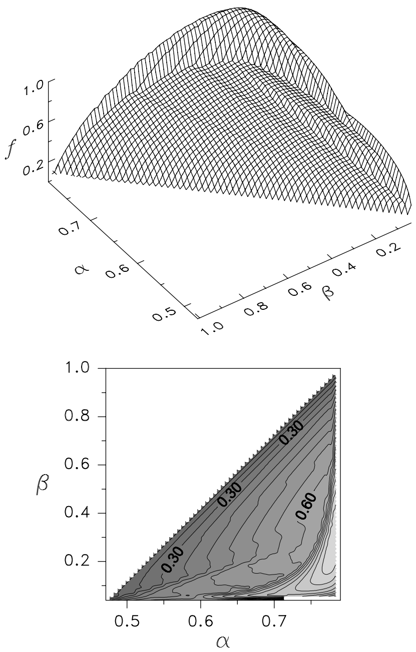

Fig. 8. Singularity spectra $f(\alpha, \beta)$ and their contour plot *vs.* α and β for $n = 100$.

5 Conclusions

The generalized multifractal formalism has been applied to strongly inhomogeneous multifractal measures with the Hölder exponent $\bar{\alpha}_{\max}$ tending to infinity as $\ell \to 0$. In contrast to the conventional multifractal formalism, the generalized one allows us to characterize such complex measures by three types (in general) of finite exponents, yielding an adequate and complete description of scaling properties of underlying local probabilities. The generalized singularity spectra are treated within the generalized formalism as functions of two or three indices. It has been shown that, due to complex structures of strongly inhomogeneous measures, their singularity spectra can exhibit very complicated forms, as compared to spectra found for typical multifractals.

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